Martingales and Rates of Presence in Homogeneous Fragmentations

Nathalie Krell, Alain Rouault

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Abstract

The main focus of this work is the asymptotic behavior of mass-conservative homogeneous fragmentations. Considering the logarithm of masses makes the situation reminiscent of branching random walks. The standard approach is to study **asymptotical** exponential rates (Berestycki [3], Bertoin and Rouault [12]). For fixed $v > 0$, either the number of fragments whose sizes at time t are of order e^{-vt} is exponentially growing with rate $C(v) > 0$, i.e. the rate is effective, or the probability of presence of such fragments is exponentially decreasing with rate $C(v) < 0$, for some concave function C. In a recent paper [21], N. Krell considered fragments whose sizes decrease at **exact** exponential rates, i.e. whose sizes are confined to be of order e^{-vs} for every $s \leq t$. In that setting, she characterized the effective rates. In the present paper we continue this analysis and focus on probabilities of presence, using the spine method and a suitable martingale. For the sake of completeness, we compare our results with those obtained in the standard approach ([3], [12]).

Key Words. Fragmentation, Lévy process, martingales, probability tilting.
A.M.S. Classification. 60J85, 65J25, 60G09.

1 Introduction and main results

We begin with a brief overview on fragmentations. We refer the reader to Bertoin [8] for a more complete exposition (and also [1] and [3]). We consider a homogenous fragmentation F of intervals, which is a Markov process in continuous time taking its values in the set $\mathcal{U}$ of open sets of $(0, 1)$. Informally, each interval component - or *fragment* - splits as time goes on, independently of the others and with the same law, up to a rescaling. We make the restriction that the fragmentation is conservative, which means that no mass is lost. In this case, the law of F is completely characterized by the so-called dislocation measure ν (corresponding to the jump-component of the process) which is a measure on $\mathcal{U}$ fulfilling the following conditions

$$\nu((0,1)) = 0,$$

$$\int_{\mathcal{U}} (1 - u_1)\nu(\mathrm{d}U) < \infty, \tag{1}$$

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and

$$\sum_{i=1}^{\infty} u_i = 1 \qquad \text{for } \nu-\text{almost every } U \ \in \mathcal{U},$$

where for $U \in \mathcal{U}$,

$$|U|^{\downarrow} := (u_1, u_2, ...)$$

is the decreasing sequence of the lengths of the interval components of U.

It is a natural question to study the rates of decay of fragments. If we measure the fragments by the logarithms of their sizes, a homogeneous fragmentation can be considered as an extension of a classical branching random walk in continuous time ([8] p. 21-22). For a broad range of branching models, the process either grows exponentially or becomes extinct. Let us recall some basic facts. If ζ_n is a Galton-Watson process started from $\zeta_0 = 1$, with finite mean $m = \mathbb{E}\zeta_1$, we have $n^{-1}\log \mathbb{E}(\zeta_n) = \log m$ and

(a) if $m > 1$ and $\mathbb{P}(\zeta_1 \geq 1) = 1$, then a.s.

$$\lim_n n^{-1} \log \zeta_n = \log m \qquad a.s.$$

(b) if $m < 1$, then

$$\exists n_0 : \forall n \geq n_0 \quad \zeta_n = 0 \qquad a.s.,$$

and

$$\lim_n n^{-1} \log \mathbb{P}(\zeta_n \neq 0) = \log m \,.$$

More generally, in a branching random walk on $\mathbb{R}$, there is a concave function λ which governs the local growth of the population. If v is some speed and Z_n is the number of particles located around nv in the n-th generation, then $\mathbb{E}Z_n$ grows exponentially at rate $\lambda(v)$. When $\lambda(v) > 0$, this quantity is also the effective exponential rate of growth of Z_n, (result of type (a), see [13]). When $\lambda(v) < 0$, a.s. Z_n is zero for n large enough and $\lambda(v)$ is the effective exponential rate of decrease of $\mathbb{P}(Z_n \neq 0)$ (result of type (b), see [26]).

The goal of this paper is to present results of the later kind (type (b)) for fragmentations, i.e. to study the asymptotic probability of presence of abnormally large fragments. Let us first explain known results of type (a) - exponential growth - and fix some notation.

For $x \in (0,1)$ let $I_x(t)$ be the component of the interval fragmentation $F(t)$ which contains x, and let $|I_x(t)|$ be its length. Bertoin showed in [6] that if V is a uniform random variable on $[0,1]$ independent of the fragmentation, then

$$\xi(t) := -\log |I_V(t)|$$

is a subordinator whose distribution is entirely determined by the characteristics of the fragmentation process. Its Laplace transform is given by

$$\mathbb{E}\mathrm{e}^{-q\xi(t)} = \mathrm{e}^{-t\kappa(q)} \quad (q \geq 0) \tag{2}$$

where κ (the Laplace exponent) is the concave function :

$$\kappa(q) := \int_{\mathcal{U}} \left(1 - \sum_{j=1}^{\infty} u_j^{q+1}\right) \nu(\mathrm{d}U)\,. \tag{3}$$

In other words,

$$\kappa(q) = \int_{(0,\infty)} \left(1 - \mathrm{e}^{-qx}\right) L(\mathrm{d}x)\,,$$

where the Lévy measure L is given by

$$L(dx) = \mathrm{e}^{-x} \sum_{j=1}^{\infty} \nu(-\log u_j \in \mathrm{d}x)\,.$$

If $\underline{p}$ is defined by

$$\underline{p} := \inf \left\{ p \in \mathbb{R} : \int_{\mathcal{U}} \sum_{j=2}^{\infty} u_j^{p+1} \nu(\mathrm{d}U) < \infty \right\}\,,$$

then Condition (1) ensures that $\underline{p} \le 0$ and we will assume throughout that $\underline{p} < 0$. It turns out that κ is an increasing concave analytic function on $(\underline{p}, \infty)$ and that (2) holds also for $q \in (\underline{p}, 0)$. Set

$$v_{\max} := \kappa'(\underline{p}^{+}) \in [0, \infty]\,.$$

The SLLN tells us that $\xi(t)/t \to \kappa'(0) =: v_{\mathrm{typ}}$ a.s., or in other words

$$\lim_{t\to\infty} -t^{-1} \log |I_V(t)| = v_{\mathrm{typ}} \quad a.s.\,.$$

To study effective exponential rates of decrease, we fix a and b two constants such that

$$0 < a < 1 < b\,.$$

In the standard approach, one considers the set of fragments

$$\widetilde{G}_{v,a,b}(t) = \{I_x(t) : x \in (0,1) \quad ae^{-vt} < |I_x(t)| < be^{-vt}\}\,.$$

To describe its behavior as $t \to \infty$, we need some notation. Define for $v < v_{\max}$

$$C(v) = \inf_{p > \underline{p}} \, (p+1)v - \kappa(p)\,. \tag{4}$$

This infimum is reached at a unique point $p = \Upsilon_v$ which is the unique solution to the equation $v = \kappa'(p)$, so that

$$C(v) := (\Upsilon_v + 1)v - \kappa(\Upsilon_v)\,. \tag{5}$$

If $\overline{p}$ is the unique solution of the equation

$$\kappa(q) = (q+1)\kappa'(q) \quad q > \underline{p}\,.$$

and if

$$v_{\min} := \kappa'(\overline{p})\,,$$

then $C(v_{\min}) = 0$, C is positive for $v \in (v_{\min}, v_{\max})$ and negative for $v < v_{\min}$. Moreover C is concave analytic and $C'(v) = \Upsilon_v + 1$ (Legendre duality).

It is known ([3], [12]) that the asymptotic growth of $\widetilde{G}_{v,a,b}(t)$ is governed by $C(v)$ (which depends only on v and not on a, b). More precisely, we have[1]:

• for $v \in (v_{\min}, v_{\max})$, then a.s.

$$\lim_{t\to\infty} t^{-1} \log \sharp \widetilde{G}_{v,a,b}(t) = C(v) \tag{6}$$

• for $v < v_{\min}$, then a.s.

$$\exists t_0 : \forall t \geq t_0 \quad \sharp \widetilde{G}_{v,a,b}(t) = 0\,.$$

In a recent paper [21], N. Krell studied the more constrained set

$$G_{v,a,b}(t) = \{I_x(t) : x \in (0,1) \text{ and } ae^{-vs} < |I_x(s)| < be^{-vs} \quad \forall\, s \leq t\}\,,$$

and proved a result of the same kind. In particular, Proposition 3 (p.908) [21] tells us that there exists a positive number $\rho(v; a, b)$ depending on v, a, b such that

• for $v > \rho(v; a, b)$, conditionally on $\{\inf\{t : G_{v,a,b}(t) = \emptyset\} = \infty\}$, a.s.

$$\lim_{t\to\infty} t^{-1} \log \sharp G_{v,a,b}(t) = v - \rho(v; a, b)\,, \tag{7}$$

• for $v < \rho(v; a, b)$, a.s.

$$\exists t_0 : \forall t \geq t_0 \quad \sharp G_{v,a,b}(t) = 0\,.$$

Since the precise definition of $\rho(v; a, b)$ is rather involved, we postpone it in the forthcoming Section 3, formula (24).

This result holds under the following assumption A, which ensures the absolute continuity of the marginals of the underlying Lévy process. Let ν_1 be the pushforward of the measure ν by the mapping $U \mapsto u_1$.

Assumption A The absolutely continuous component ν_1^{ac} of ν_1 with respect to the Lebesgue measure on $[0, 1]$ satisfies

$$\nu_1^{\mathrm{ac}}((1-\epsilon, 1]) = \infty \quad \text{for any} \quad \epsilon > 0\,. \tag{8}$$

The study of $\widetilde{G}_{v,a,b}$ or $G_{v,a,b}$ will be referred as the *classical* or *confined* model, respectively. According to the above informal classification of results on branching models, we can say that the above assertions (6) and (7) are of type (a) on page 2. Our objective here is to present results of type (b) on page 2.

For the *classical* model, an assumption is needed. A fragmentation is called r-lattice with $r > 0$, if $\xi(t)$ is a compound Poisson process whose jump measure has a support carried by a discrete subgroup of $\mathbb{R}$ and r is the mesh. If there is no such r, the fragmentation is called non-lattice.

Assumption B. Either the fragmentation is non-lattice, or it is r-lattice and a, b satisfy $b > a\mathrm{e}^r$.

Theorem 1.1. *[11] Under Assumption B, if $v < v_{\min}$, then*

$$\lim_{t\to\infty} t^{-1} \log \mathbb{P}(\widetilde{G}_{v,a,b}(t) \neq \emptyset) = C(v)\,. \tag{9}$$

[1] $\sharp A$ stands for the cardinality of the set A

In [11], the result of Theorem 5 is more precise since it gives sharp (i.e. non logarithmic) estimates of the latter probability.

For the more constrained set $G_{v,a,b}(t)$, we have the following theorem, which is the main result of the present paper.

Theorem 1.2. *Under Assumption A, if $v - \rho(v; a, b) < 0$, then*

$$\lim_{t\to\infty} t^{-1} \log \mathbb{P}(G_{v,a,b}(t) \neq \emptyset) = v - \rho(v; a, b)\,. \tag{10}$$

Let us remark that since $G_{v,a,b} \subset \tilde{G}_{v,a,b}$, the limits (7) and (10) are smaller than the limits (6) and (9), respectively. In fact we have the following general result, which extends Remark 4 in [21].

Proposition 1.3. *For all $v < v_{\max}$, then*

$$C(v) \geq v - \rho(v; a, b). \tag{11}$$

Let us explain shortly our method, whose crucial tools were already central in the proof of results of type (a) in [21], namely the construction of an additive martingale, the corresponding change of probability and the so-called spine decomposition.

In branching or fragmentation problems, it is by now customary to enlarge the probability space by considering a randomly chosen branch or a randomly tagged fragment, respectively. This random element is called the "spine". Informally, we deal with two filtrations : a large one including the spine and a small one without the spine. A martingale built on the observation of the spine process may be projected on the small filtration, obtaining a so-called "additive" martingale. These martingales induce changes of probability. Making use of a proper choice of the martingale measurable with respect to the large filtration, the spine has generally a nice behavior under this new probability whereas the behavior of the other particles (or fragments) is not affected by this change. It is then possible to split the additive martingale into two parts : the contribution of the spine and the contribution of other terms (this is called the spine decomposition). It allows to describe in the large time limit the behavior of the additive martingale itself and the behavior of the branching or fragmentation.

For the *classical* model, the Esscher martingale is convenient to study $\widetilde{G}_{v,a,b}$ (see [12]). For the branching random walk and related processes, a good recent reference with historical comments is [17]. The change of probability forces the tagged fragment to have a "good" asymptotic logarithmic rate of decrease.

For the *constrained* model, as in [21], we have been inspired by the change of probability introduced by Bertoin [5] and Lambert [25]. It has the effect of forcing the spine to be confined in some interval, as required to study $G_{v,a,b}$.

In Section 2, we summarize the basic notions on fragmentations and Lévy processes which will be needed later. Section 3 is devoted to the study of the two martingales and their asymptotic properties. In Section 4, we give the proofs of the theorems on the presence probabilities and the proofs of the results on martingales[2]. For the sake of completeness, a direct short proof of Theorem 1.1 with the spine method is given to illustrate the common feature of both models (it should be noted that a similar method was applied to obtain analogous results in the context of branching Brownian motion in [18]). Section 5 is devoted to a proof of Proposition 1.3, only based on properties of Lévy processes.

[2]In particular, a mistake in the proof of Theorem 2.1 in [21] is corrected.

2 Background on fragmentations and Lévy processes.

2.1 Partition fragmentations and interval fragmentations

Let $\mathbb{N}$ stand for the set of positive integers; a block is a subset of $\mathbb{N}$. For every $k \in \mathbb{N}$, the block $\{1, ..., k\}$ is denoted by $[k]$. Let $\mathcal{P}$ the space of partitions of $\mathbb{N}$. As in [10], any measure:

$$\omega = \sum_{(t,\pi,k)\in\mathcal{D}}^{\infty} \delta_{(t,\pi,k)},$$

where $\mathcal{D}$ is a subset of $\mathbb{R}_+ \times \mathcal{P} \times \mathbb{N}$ such that

$$\forall t^{'} \geq 0 \;\; \forall n \in \mathbb{N} \;\; \sharp\Big\{(t, \pi, k) \in \mathcal{D} \mid t \leq t^{'}, \pi_{|[n]} \neq ([n], \emptyset, \emptyset, ...), k \leq n\Big\} < \infty \tag{12}$$

and for all $t \in \mathbb{R}$

$$\omega(\{t\} \times \mathcal{P} \times \mathbb{N}) \in \{0, 1\}.$$

is called a discrete point measure on the space $\Omega := \mathbb{R}_+ \times \mathcal{P} \times \mathbb{N}$. Starting from an arbitrary discrete point measure ω on $\mathbb{R}_+ \times \mathcal{P} \times \mathbb{N}$, we will construct a nested partition $\Pi = (\Pi(t), t \geq 0)$ (which means that for all $t \geq t^{'}$ $\Pi(t)$ is a finer partition of $\mathbb{N}$ than $\Pi(t^{'})$). We fix $n \in \mathbb{N}$, the assumption (12) that the point measure ω is discrete enables us to construct a step path $(\Pi(t, n), t \geq 0)$ with values in the space of partitions of $[n]$, which only jumps at times t at which the fiber $\{t\} \times \mathcal{P} \times \mathbb{N}$ carries an atom of ω, say (t, π, k), such that $\pi_{|[n]} \neq ([n], \emptyset, \emptyset, ...)$ and $k \leq n$. In that case, $\Pi(t, n)$ is the partition obtained by replacing the k-th block of $\Pi(t-, n)$, denoted $\Pi_k(t-, n)$, by the restriction $\pi_{|\Pi_k(t-,n)}$ of π to this block, and leaving the other blocks unchanged. Of course for all $t \geq 0$, $(\Pi(t, n), n \geq 0)$ is compatible (i.e. for every n, $\Pi(n, t)$ is a partition of $[n]$ such that the restriction of $\Pi(n+1, t)$ to $[n]$ coincide with $\Pi(n, t)$). As a consequence, there exists a unique partition $\Pi(t)$, such that for all $n \geq 0$ we have $\Pi(t)_{|[n]} = \Pi(t, n)$. This process Π is a partition-valued homogeneous fragmentation ([8] chap. 3).

Let the set $\mathcal{S}^{\downarrow}$ be

$$\mathcal{S}^{\downarrow} := \left\{ s = (s_1, s_2, ...) \mid s_1 \geq s_2 \geq ... \geq 0 \;, \sum_{i=1}^{\infty} s_i \leq 1 \right\}.$$

A block B has an asymptotic frequency, if the limit

$$|B| := \lim_{n\to\infty} n^{-1} \sharp(B \cap [n])$$

exists. When every block of some partition $\pi \in \mathcal{P}$ has an asymptotic frequency, we write $|\pi| = (|\pi_1|, ...)$ and then $|\pi|^{\downarrow} = (|\pi_1|^{\downarrow}, ...) \in \mathcal{S}^{\downarrow}$ for the decreasing rearrangement of the sequence $|\pi|$. If a block of the partition π does not have an asymptotic frequency, we decide that $|\pi| = |\pi|^{\downarrow} = \partial$, where ∂ stands for some extra point added to $\mathcal{S}^{\downarrow}$.

On Ω, the sigma-field generated by the restriction to $[0, t] \times \mathcal{P} \times \mathbb{N}$ is denoted by $\mathcal{G}(t)$. So $\mathcal{G} = (\mathcal{G}(t))_{t\geq 0}$ is a filtration, and the nested partitions $(\Pi(t), t \geq 0)$ are $\mathcal{G}$-adapted. If $|\Pi(r)|^{\downarrow}$ is the decreasing rearrangement of the sequence of the asymptotic frequencies of the blocks of $\Pi(r)$, we denote by $\mathcal{F}(t)$ the sigma-field generated by $\left(|\Pi(r)|^{\downarrow}\right)_{r\leq t}$. Of course $\mathcal{F} = (\mathcal{F}(t))_{t\geq 0}$ is a sub-filtration of $\mathcal{G}$.

Let $\mathcal{G}_1(t)$ the sigma-field generated by the restriction of the discrete point measure ω to the fiber $[0,t]\times\mathcal{P}\times\{1\}$. So $\mathcal{G}_1 = (\mathcal{G}_1(t))_{t\geq 0}$ is a sub-filtration of $\mathcal{G}$, and the first block of Π is $\mathcal{G}_1$-measurable. Let $\mathcal{D}_1 \subseteq \mathbb{R}_+$ be the random set of times $r \geq 0$ for which the discrete point measure has an atom on the fiber $\{r\}\times\mathcal{P}\times\{1\}$, and for every $r \in \mathcal{D}_1$, denote the second component of this atom by $\pi(r)$.

There is a powerful link between partition fragmentations and interval fragmentations. On the one hand, the $\mathcal{S}^{\downarrow}$-valued process of ranked asymptotic frequencies $|\Pi|^{\downarrow}$ of a partition fragmentation is a so-called ranked (or mass) fragmentation ([2], [6]), and conversely a partition fragmentation can be built from a ranked fragmentation via a "paintbox" process. On the other hand, the interval decomposition $(J_i(t), J_2(t), ...)$ of the open $F(t)$ ranked in decreasing order is a ranked fragmentation, denoted by $X(t) := (|J_i(t)|, |J_2(t)|, ...)^{\downarrow}$. We can then lift this ranked fragmentation to a partition fragmentation. More precisely, if ν is the dislocation measure of F, and $\tilde{\nu}$ its image by the map $U \mapsto |U|^{\downarrow}$, then according to Theorem 2 in [6], there exists a unique measure μ on $\mathcal{P}$ which is exchangeable (i.e. invariant by the action of finite permutations on $\mathcal{P}$), and such that $\tilde{\nu}$ is the image of μ by the map $\pi \mapsto |\pi|^{\downarrow}$ where $|\pi|^{\downarrow}$ is the decreasing rearrangement of the sequence of the asymptotic frequencies of the blocks of π. So, for all measurable function $f : [0,1] \to \mathbb{R}_+$ such that $f(0) = 0$,

$$\int_{\mathcal{P}} f(|\pi_1|)\mu(\mathrm{d}\pi) = \int_{\mathcal{S}^{\downarrow}} \sum_{i=1}^{\infty} s_i f(s_i)\tilde{\nu}(\mathrm{d}s) = \int_{\mathcal{U}} \sum_{i=1}^{\infty} u_i f(u_i)\nu(\mathrm{d}U)\,.$$

It should be noted that $\{|J_1(t)|, |J_2(t)|, ...\}_{t\geq 0} = \{|\Pi_1(t)|, |\Pi_2(t)|, ...\}_{t\geq 0}$ (equality in distribution in general, and true equality if Π is obtained by a paintbox process).

In the following sections, Π refers to this partition fragmentation.

2.2 Lévy processes.

A Lévy process is a stochastic process with càdlàg sample paths and stationary independent increments ([4]). In this work, two types of such processes will play a key role :

• a subordinator is a Lévy process taking values in $[0,\infty)$, which implies that its sample paths are increasing,

• a Lévy process is completely asymmetric when its jumps are either all positive or all negative. We will consider here Lévy processes without positive jumps, i.e. spectrally negative processes.

The Laplace transform of a subordinator $\sigma = (\sigma_t)_{t\geq 0}$ is given by [3] :

$$\mathbf{E} \exp -\lambda\sigma_t = \exp -t\Phi(\lambda), \quad \lambda \geq 0\,, \tag{13}$$

where Φ is called the Laplace exponent. If $\mathcal{E} = (\mathcal{E}_t)_{t\geq 0}$ is the natural filtration associated with σ

$$(\exp(-p\sigma_t + t\Phi(p)))_{t\geq 0}$$

is a $\mathcal{E}$-martingale. We define the probability measure $\mathbf{P}^{(p)}$ as the Esscher transform:

$$\mathrm{d}\mathbf{P}^{(p)}|_{\mathcal{E}_t} = \exp\{-p\sigma_t + t\Phi(p)\}\ \mathrm{d}\mathbf{P}|_{\mathcal{E}_t}\,. \tag{14}$$

[3]Bold symbols $\mathbf{P}$ and $\mathbf{E}$ will refer to Lévy processes while $\mathbb{P}$ and $\mathbb{E}$ refer to fragmentations.

Under $\mathbf{P}^{(p)}$, σ is a subordinator with Laplace exponent $q \mapsto \Phi(p+q) - \Phi(p)$. The change of probability forces the process to have mean $t\Phi'(p)$ at time t. It also exponentially tilts the Lévy measure.

Let us recall some facts about about completely asymmetric Lévy processes, lifted from [4] and [5]. Let $Y = (Y_t)_{t\geq 0}$ be a Lévy process with no positive jumps and let $\mathcal{E} = (\mathcal{E}_t)_{t\geq 0}$ be the natural filtration associated with Y. The case where Y is the negative of a subordinator is degenerate for our purpose and is therefore implicitly excluded in the rest of the paper. The law of the Lévy process started at $x \in \mathbb{R}$ will be denoted by $\mathbf{P}_x$, its Laplace transform is given by

$$\mathbf{E}_0(\mathrm{e}^{\lambda Y_t}) = \mathrm{e}^{t\psi(\lambda)}, \quad \lambda \geq 0,$$

where $\psi : \mathbb{R}_+ \to \mathbb{R}$ is called the Laplace exponent. The function ψ is convex with $\lim_{\lambda\to\infty} \psi(\lambda) = \infty$.

Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be the right inverse of ψ so that $\psi(\phi(\lambda)) = \lambda$ for every $\lambda \geq 0$. The scale function $W : \mathbb{R}_+ \to \mathbb{R}_+$ is the unique continuous function with Laplace transform:

$$\int_0^\infty e^{-\lambda x} W(x)\mathrm{d}x = \frac{1}{\psi(\lambda)} \quad , \ \lambda > \phi(0).$$

For $q \in \mathbb{R}$, let $W^{(q)} : \mathbb{R}_+ \to \mathbb{R}$ be the continuous function defined by

$$W^{(q)}(x) := \sum_{k=0}^{\infty} q^k W^{*k+1}(x)\,,$$

where W^{*k} is the k-th convolution of W with himself, so that

$$\int_0^\infty e^{-\lambda x} W^{(q)}(x)\mathrm{d}x = \frac{1}{\psi(\lambda) - q} \quad , \ \lambda > \phi(q)\,.$$

For fixed x, $W^{(q)}(x)$ can be seen as an analytical function in q. The functions $W^{(q)}$ are useful to investigate the two-sided exit problem for Lévy processes. Their properties are well exposed in the book of Kyprianou [23] and in [14], examples are in [19] and in [24].

The following theorems 2.1 and 2.3 taken from [25] and [5] yield another important martingale and its corresponding change of probability.

Theorem 2.1. *Suppose that the one-dimensional distributions of the Lévy process Y are absolutely continuous. Let us define the critical value*

$$\rho_\beta := \inf\{q \geq 0 \ ; \ W^{(-q)}(\beta) = 0\}. \tag{15}$$

Then the following holds:

1. *$\rho_\beta \in (0,\infty)$ and the function $W^{(-\rho_\beta)}$ is strictly positive on $(0,\beta)$.*
2. *For $\beta > 0$, let*
$$T_\beta = \inf\{t : \ Y_t \notin (0,\beta)\} \ ; \tag{16}$$
then the process $D = (D_t)_{t\geq 0}$ with
$$D_t := e^{\rho_\beta t} \ \mathbf{1}_{\{t<T_\beta\}} \ \frac{W^{(-\rho_\beta)}(Y_t)}{W^{(-\rho_\beta)}(x)} \tag{17}$$
is a $(\mathbf{P}_x, \mathcal{E})$-martingale, for every $x \in (0,\beta)$.

3. *The mapping $\beta \mapsto \rho_\beta$ is strictly decreasing and continuous on $(0, \infty)$.*

Point 1 is taken from [5] Theorem 2(i) and (iii). Point 2 is from [25] Theorem 3.1 (ii). Point 3 is from [25] Proposition 5.1 (ii).

Remark 2.2. *1. Notice that Proposition 5.1 in [25], devoted to the smoothness properties of functions W and ρ, is claimed for paths with unbounded variation or with bounded variation provided the Lévy measure has no atoms. However, this additional assumption is not used in the part of the proof dedicated to our point 3, it is used to prove stronger regularity. However we do not need this assumption, since we only care about continuity. Let us stress that the smoothness of scale functions is a very active subject, see [14].*

2. The definition of ρ_β is complicated, but some examples are given in [21].

Let the probability measure $\mathbf{P}^{\updownarrow}$ be the h-transform of $\mathbf{P}$ based on the martingale D :

$$\mathrm{d}\mathbf{P}_x^{\updownarrow}|_{\mathcal{E}_t} = D_t \, \mathrm{d}\mathbf{P}_x|_{\mathcal{E}_t}. \tag{18}$$

Theorem 2.3. *With the same assumption as in Theorem 2.1, under $\mathbf{P}_x^{\updownarrow}$, Y is a homogeneous strong Markov process on $(0, \beta)$, positive-recurrent and as $t \to \infty$, Y_t converges in distribution to its stationary probability, which has a density.*

This is essentially a rephrasing of Theorem 3.1 in [25], the convergence in distribution is a consequence of Theorem 2 (v) of [5].

The change of probability forces the process to be confined in $(0, \beta)$. In the Brownian motion case, this is sometimes called a taboo process [20].

3 Two additive martingales and their asymptotic behavior

One of the most striking fact about homogeneous fragmentations is the subordinator representation. If $\xi_t = -\ln |\Pi_1(t)|$, then, as seen in [8] (Section 3.2.2), the process $\xi = (\xi_t)_{t \geq 0}$ defined on $(\Omega, \mathcal{G}, \mathbb{P})$ is a subordinator, which means in particular that $\xi_{t+s} - \xi_t$ is independent of $\mathcal{G}(t)$. In this section, we will adapt the statements of Section 2.2 to the subordinator ξ (instead of σ) and to the spectrally negative process $Y = (Y_t = vt - \xi_t - \log a)_{t \geq 0}$, starting at $x = -\log a$. It should be stressed that $\mathcal{G}$ is not the proper filtration of these processes, but the martingale properties remain true, as well as the Markov property. We will then perform a projection on the filtration $\mathcal{F}$ of the ranked fragmentation.

3.1 The classical additive martingale $M^{(p)}$

As in (14), we define for $p > \underline{p}$ the $\mathcal{G}$-martingale $D^{(p)} = \left(D_t^{(p)}\right)_{t \geq 0}$ where

$$D_t^{(p)} = \mathrm{e}^{-p\xi(t) + t\kappa(p)},$$

and the probability measure $\mathbb{P}^{(p)}$ as the transform :

$$\mathrm{d}\mathbb{P}^{(p)}|_{\mathcal{G}(t)} = D_t^{(p)} \mathrm{d}\mathbb{P}|_{\mathcal{G}(t)}. \tag{19}$$

Projecting the martingale $D^{(p)}$ on the sub-filtration $\mathcal{F}$, we obtain the well-known additive $\mathcal{F}$-martingale $M^{(p)} = \left(M_t^{(p)}\right)_{t\geq 0}$, where

$$M_t^{(p)} = \sum_{j=1}^{\infty} |\Pi_j(t)|^{p+1} e^{\kappa(p)t} = \sum_{i=1}^{\infty} |J_i(t)|^{p+1} e^{\kappa(p)t} \,. \tag{20}$$

The projection of (19)
gives the identity:

$$\mathrm{d}\mathbb{P}^{(p)}|_{\mathcal{F}(t)} \;= M_t^{(p)} \mathrm{d}\mathbb{P}|_{\mathcal{F}(t)} \,. \tag{21}$$

In [10] Proposition 6, there is a complete description of the behavior of the process Π. We keep in mind the next result.

Lemma 3.1. *Under $\mathbb{P}^{(p)}$, the process ξ is a subordinator with Laplace exponent $q \mapsto \kappa(p+q) - \kappa(p)$.*

3.2 The martingale $M^{(v,a,b)}$ and the confined process.

Since we are interested in the set of the "good intervals" at time t as

$$G_{v,a,b}(t) = \left\{ I_x(t) : \; x \in (0,1) \;\text{ and }\; ae^{-vs} < |I_x(s)| < be^{-vs} \;\; \forall \; s \leq t \right\} \tag{22}$$

it is natural to study the Lévy process $Y = (Y_t)_{t\geq 0}$ defined by

$$Y_t := vt - \xi_t - \log a$$

and its exit time from $(0, \log b/a)$. Clearly Y has no positive jump and its Laplace exponent is

$$\psi(\lambda) = v\lambda - \kappa(\lambda) \,, \tag{23}$$

with κ defined in (3).

Assumption (A) guarantees that the marginals of Y are absolutely continuous and Theorem 2.1 can thus be applied. Let us fix some notation. The distribution of Y depends on v and we set

$$\rho(v; a, b) := \rho_{\log(b/a)} \;,\; T := T_{\log(b/a)} \tag{24}$$

where ρ_β and T_β are defined in (15) and (16), respectively. We will use frequently the notation ρ instead of $\rho(v; a, b)$.

To further simplify the notation, let also

$$h(y) := W^{(-\rho)}(y - \log a) \mathbf{1}_{\{y \in (\log a, \log b)\}} \tag{25}$$

for all $y \in \mathbb{R}$, and $h(-\infty) = 0$. This function is well defined thanks to Theorem 2.1 and $h(0) \neq 0$.

By translating (17) into the new notation we get a $\mathcal{G}$-martingale $D = (D_t)_{t\geq 0}$

$$D_t = \mathrm{e}^{\rho t} \; \mathbf{1}_{\{t<T\}} \, \frac{h(vt - \xi_t)}{h(0)} \,, \tag{26}$$

and then a new probability defined by

$$\mathrm{d}\mathbb{P}^{\updownarrow}|_{\mathcal{G}(t)} \;= D_t \, \mathrm{d}\mathbb{P}|_{\mathcal{G}(t)} \,. \tag{27}$$

For $i \geq 1$, let $P_i(t)$ be the block of $\Pi(t)$ which contains i at time t. We define the killed partition as follows

$$\Pi_j^{\dagger}(t) = \Pi_j(t)\mathbf{1}_{\{\exists i\in\mathbb{N}^* |\ \Pi_j(t)=P_i(t);\ \forall s\leq t\quad |P_i(s)|\in(ae^{-vs},be^{-vs})\}}.$$

Similarly, if I is an interval component of $F(t)$, we define the "killed" interval $I^{\dagger}$ by $I^{\dagger} = I$ if I is good (i.e. $I \in G_{v,a,b}(t)$ with $G_{v,a,b}(t)$ defined in (22)), else by $I^{\dagger} = \emptyset$. Projecting the martingale D on the sub-filtration $\mathcal{F}$, we obtain an additive martingale $M^{(v,a,b)} = \left(M_t^{(v,a,b)}\right)_{t\geq 0}$ where

$$\begin{aligned} M_t^{(v,a,b)} &= \frac{e^{\rho t}}{h(0)} \sum_{j\in\mathbb{N}} h\left(vt + \log|\Pi_j^{\dagger}(t)|\right) |\Pi_j^{\dagger}(t)| \\ &= \frac{e^{\rho t}}{h(0)} \sum_{i\in\mathbb{N}} h\left(vt + \log|J_i^{\dagger}(t)|\right) |J_i^{\dagger}(t)|\,. \end{aligned}$$

Finally, let the absorption time of $M^{(v,a,b)}$ at 0 be

$$\zeta := \inf\{t : M_t^{(v,a,b)} = 0\} = \inf\{t : G_{v,a,b}(t) = \emptyset\},$$

with the convention $\inf\emptyset = \infty$.

The projection of (27) on $\mathcal{F}$ gives the identity:

$$\mathrm{d}\mathbb{P}^{\updownarrow}|_{\mathcal{F}(t)} = M_t^{(v,a,b)}\mathrm{d}\mathbb{P}|_{\mathcal{F}(t)}\,. \tag{28}$$

The upshot is that the change of probability $\mathbb{P}^{\updownarrow}$ only affects the behavior of the block which contains 1. More precisely, like in lemma 8 (ii) [12], we obtain:

Lemma 3.2. *Suppose Assumption (A) holds. Under $\mathbb{P}^{\updownarrow}$, the restriction of ω to $\mathbb{R}_+ \times \mathcal{P} \times \{2, 3, ...\}$ has the same distribution as under $\mathbb{P}$.*

3.3 Growth of martingales

The next theorems govern the asymptotic behaviors of our martingales $M^{(v,a,b)}$ and $M^{(p)}$, according to the values of parameters v and p.

It should be noted that assertion 1 of Theorem 3.3 was claimed in [21] Theorem 2, but unfortunately there was a mistake in the proof. Indeed it is not true in general that the function h defined in (25) is Lipschitz at 0. See again Remark 2.2 for smoothness of W (and hence of h).

The points 1) and 2 a) of Theorem 3.4 are known ([7] p. 406-407 and [10] respectively). We will recall the argument for the sake of completeness.

Theorem 3.3. *Suppose assumption A holds, then:*

1. *If $v > \rho(v; a, b)$, the martingale $M^{(v,a,b)}$ is bounded in $\mathrm{L}^2(\mathbb{P})$.*
2. *If $v < \rho(v; a, b)$,*
   a) $\lim_{t\to\infty} M_t^{(v,a,b)} = 0$, $\mathbb{P}$-a.s.,
   b) there exists $K_1, K_2 > 0$ such that for every t

$$K_1 \leq \mathrm{e}^{(v-\rho(v;a,b))t}\,\mathbb{E}\left[M_t^{(v,a,b)}\right]^2 \leq K_2\,. \tag{29}$$

Theorem 3.4. *1. If $p \in (\underline{p}, \overline{p})$, there exists $\alpha > 0$ such that the martingale $M^{(p)}$ is bounded in $\mathrm{L}^{1+\alpha}(\mathbb{P})$.*

2. If $p \geq \overline{p}$,

a) $\lim_{t\to\infty} M_t^{(p)} = 0$, $\mathbb{P}$-a.s.

b) There exists $\alpha_0 > 0$ such that for $\alpha \in (0, \alpha_0)$,

$$d(p,\alpha) := (1+\alpha)\kappa(p) - \kappa\left((1+\alpha)(p+1) - 1\right) > 0$$

and then for those α, we have for every $t > 0$

$$\mathrm{e}^{d(p,\alpha)t}\, \mathbb{E}\, |M_t^{(p)}|^{1+\alpha} \leq C_{\alpha,p}\,, \tag{30}$$

where $C_{\alpha,p}$ depends on α and p.

4 Proofs

4.1 Proof of Theorem 1.2

Proof: • We first show the upper bound of (10), i.e.

$$\limsup_{t\to\infty}\ t^{-1} \log \mathbb{P}(G_{v,a,b}(t) \neq \emptyset) \leq v - \rho(v;a,b)\,. \tag{31}$$

Let $0 < \bar{a} < a < 1 < b < \bar{b}$. As in Section 3.2, we associate with $\bar{a}, \bar{b}$ and v, the parameter $\bar{\rho} = \rho(v;\bar{a},\bar{b})$, as well as the set of "good" intervals

$$\bar{G}(t) = G_{v,\bar{a},\bar{b}}(t) := \{I_x(t) : \quad x \in (0,1) \quad \text{and} \quad |I_x(s)| \ \in\ (\bar{a}e^{-vs}, \bar{b}e^{-vs}) \quad \forall\ s \leq t\}\,,$$

and the martingale $\bar{M} = M^{v,\bar{a},\bar{b}}$.

Let for every $y \in \mathbb{R}$

$$\bar{h}(y) := W^{(-\bar{\rho})}(y - \log \bar{a})\mathbf{1}_{\{y\in(\log \bar{a}, \log \bar{b})\}}\,.$$

For $t \geq 0$ fixed, we have:

$$\begin{aligned} 1 = \mathbb{E}\bar{M}_t &= \frac{e^{\bar{\rho}t}}{\bar{h}(0)}\mathbb{E}\left(\sum_{i\in\mathbb{N}} \bar{h}(vt + \log|J_i(t)|)|J_i(t)|\ \mathbf{1}_{\{J_i(t)\in\bar{G}(t)\}}\right) \\ &\geq \frac{\bar{a}e^{(\bar{\rho}-v)t}}{\bar{h}(0)}\mathbb{E}\left(\sum_{i\in\mathbb{N}} \bar{h}(vt + \log|J_i(t)|)\ \ \mathbf{1}_{\{J_i(t)\in G_{v,a,b}(t)\}}\right)\,. \end{aligned}$$

Since $(a,b) \subsetneq (\bar{a},\bar{b})$, the function $\bar{h}$ is continuous and strictly positive on $[\log a, \log b]$ so that, if

$$K_3 := \bar{h}(0)/\left(\bar{a} \inf_{x\in[\log a,\log b]} \bar{h}(x)\right) < \infty,$$

then, for all $t \geq 0$:

$$K_3 \geq e^{(\bar{\rho}-v)t}\mathbb{E}\left(\sum_{i\in\mathbb{N}} \mathbf{1}_{\{J_i(t)\in G_{v,a,b}(t)\}}\right) \geq e^{(\bar{\rho}-v)t}\mathbb{P}(G_{v,a,b}(t) \neq \emptyset)\,,$$

and consequently

$$\limsup_{t\to\infty}\ t^{-1}\log \mathbb{P}(G_{v,a,b}(t) \neq 0) \leq v - \bar{\rho} = v - \rho(v;\bar{a},\bar{b})\,.$$

Since it holds true for all $\bar{a}, \bar{b}$ such that $0 < \bar{a} < a < 1 < b < \bar{b}$, we can let $\bar{a} \to a$ and $\bar{b} \to b$ and use the continuity of $\rho(v;\cdot,\cdot)$ (see Theorem 2.1.3) to obtain the inequality (31).

• It remains to prove the lower bound of (10), i.e.

$$\liminf_{t\to\infty}\ t^{-1}\log \mathbb{P}(G_{v,a,b}(t) \neq \emptyset) \geq v - \rho\,. \tag{32}$$

Let us drop the subscripts and superscripts (v, a, b). Since M is a positive martingale and $\{G(t) \neq \emptyset\} = \{M_t \neq 0\}$, we have

$$1 = \mathbb{E}(M_t) = \mathbb{E}(M_t \mathbb{1}_{\{M_t\neq 0\}}) = \mathbb{E}(M_t \mathbb{1}_{\{G(t)\neq\emptyset\}})\,.$$

Now, thanks to the Cauchy-Schwarz inequality:

$$\mathbb{E}(M_t \mathbb{1}_{\{G(t)\neq\emptyset\}}) \leq \left(\mathbb{E}(M_t^2)\right)^{1/2} \left(\mathbb{P}(G(t) \neq \emptyset)\right)^{1/2}$$

and applying (29), we get

$$\mathbb{P}(G(t) \neq \emptyset) \geq K_2^{-1} e^{(v-\rho)t}\,,$$

which yields (32). ■

4.2 Proof of Theorem 1.1

The upper bound is straightforward. For all $p \geq \underline{p}$, we have

$$1 = \mathbb{E}M_t^{(p)} = \mathbb{E}\left(\sum_{i=1}^{\infty} |J_i(t)|^{p+1} \mathrm{e}^{\kappa(p)t}\right) \geq a^{p+1}\mathrm{e}^{\kappa(p)t-(p+1)vt}\mathbb{P}(\widetilde{G}_{v,a,b}(t) \neq \emptyset)\,.$$

Hence

$$\mathbb{P}(\widetilde{G}_{v,a,b}(t) \neq \emptyset) \leq a^{-(p+1)}\mathrm{e}^{[(p+1)v-\kappa(p)]t}$$

and

$$\limsup_{t\to\infty}\ t^{-1}\log \mathbb{P}(\widetilde{G}_{v,a,b}(t) \neq \emptyset) \leq (p+1)v - \kappa(p).$$

In particular, for $p = \Upsilon_v$, we get from (5)

$$\limsup_{t\to\infty}\ t^{-1}\log \mathbb{P}(\widetilde{G}_{v,a,b}(t) \neq \emptyset) \leq C(v)\,.$$

To prove the lower bound

$$\liminf_{t\to\infty}\ t^{-1}\log\mathbb{P}(\widetilde{G}_{v,a,b}(t)\neq\emptyset)\geq C(v)\,, \tag{33}$$

we use again the change of probability (21) with $p=\Upsilon_v$. We have,

$$\mathbb{P}(\widetilde{G}_{v,a,b}(t)\neq\emptyset)=\mathbb{E}^{(p)}\left((M_t^{(p)})^{-1};\widetilde{G}_{v,a,b}(t)\neq\emptyset\right)\geq$$

$$\mathrm{e}^{tC(v)-t\varepsilon}\,\mathbb{P}^{(p)}\left(\sup_{0<s\leq t}M_s^{(p)}<\mathrm{e}^{-tC(v)+t\varepsilon};vt-\xi_t\in[\log a,\log b]\right) \tag{34}$$

and

$$\mathbb{P}^{(p)}\left(\sup_{0<s\leq t}M_s^{(p)}<\mathrm{e}^{-tC(v)+t\varepsilon};vt-\xi_t\in[\log a,\log b]\right)\geq$$

$$\mathbb{P}^{(p)}\left(vt-\xi_t\in[\log a,\log b]\right)-\mathbb{P}^{(p)}\left(\sup_{0<s\leq t}M_s^{(p)}\geq\mathrm{e}^{-tC(v)+t\varepsilon}\right). \tag{35}$$

From Lemma 3.1 we see that under $\mathbb{P}^{(p)}$, the Lévy process $(vt-\xi_t)_{t\geq0}$ has mean $-\kappa'(p)+v=0$ and variance $\sigma_p^2:=-\kappa''(p)$. From Proposition 2 of Bertoin and Doney [9] it satisfies the local central limit theorem, if it is not lattice. We get

$$\sigma_p\sqrt{2\pi t}\,\mathbb{P}^{(p)}(vt-\xi_t\in[\log a,\log b])\to\log\frac{b}{a}\,. \tag{36}$$

and then

$$\liminf_t\ t^{-1}\log\mathbb{P}^{(p)}(vt-\xi_t\in[\log a,\log b])=0\,. \tag{37}$$

In the case of a r-lattice fragmentation, under assumption B, there is at least an integer multiple of r in the interval $[vt-\log b,vt-\log a]$. We can use the lattice version of the local central limit theorem (see for instance [15] Theorem 2 iii)), to obtain (37) again.

To tackle the second term of the RHS of (35), we argue as in [18]. By convexity $((M_t^{(p)})^{1+\alpha},t\geq 0)$ is a $\mathbb{P}$-submartingale, so $((M_t^{(p)})^{\alpha}_{t\geq0},t\geq0)$ is a $\mathbb{P}^{(p)}$-submartingale. Hence, by Doob's inequality,

$$\begin{aligned}\mathbb{P}^{(p)}\left(\sup_{0<s\leq t}|\,M_s^{(p)}\,|\geq\mathrm{e}^{-tC(v)+t\varepsilon}\right)&\leq\mathrm{e}^{t\alpha C(v)-\alpha t\varepsilon}\mathbb{E}^{(p)}\,|\,M_t^{(p)}\,|^{\alpha}\\&=\mathrm{e}^{t\alpha C(v)-\alpha t\varepsilon}\mathbb{E}\,|\,M_t^{(p)}\,|^{1+\alpha}\,,\end{aligned} \tag{38}$$

and by (30) we have for $\alpha\in(0,\alpha_1]$ for some $\alpha_1>0$

$$\mathbb{P}^{(p)}\left(\sup_{0<s\leq t}|\,M_s^{(p)}\,|\geq\mathrm{e}^{-tC(v)+t\varepsilon}\right)\leq K'_{\alpha,p}\mathrm{e}^{tH(\alpha)}\,, \tag{39}$$

where

$$H(\alpha)=\alpha C(v)-\alpha\varepsilon+d(p,\alpha)\,,$$

and $K'_{\alpha,p}$ is some constant. Now, a second order development of κ around p gives

$$H(\alpha) = -\alpha\varepsilon - \frac{\alpha^2(p+1)^2}{2}\kappa''(p)(1+o(\alpha))$$

and, since $\kappa'' < 0$ (κ is concave), we may choose α small enough such that $H(\alpha) < 0$. This yields

$$\limsup_t \ t^{-1} \log \mathbb{P}^{(p)} \left(\sup_{0<s\leq t} |M_s^{(p)}| \geq \mathrm{e}^{-tC(v)+t\varepsilon} \right) < 0. \tag{40}$$

So, gathering (40) and (36), we obtain

$$\liminf_{t\to\infty} t^{-1} \log \mathbb{P}^{(p)} \left(\sup_{0<s\leq t} M_s^{(p)} < \mathrm{e}^{-tC(v)+t\varepsilon}; vt - \xi_t \in [\log a, \log b] \right) = 0\,,$$

which, with (34), yields

$$\liminf_{t\to\infty} t^{-1} \log \mathbb{P}(\widetilde{G}_{v,a,b}(t) \neq \emptyset) \geq C(v) - \varepsilon$$

for all $\varepsilon > 0$. Letting $\varepsilon \to 0$ proves (33), and concludes the proof of Theorem 1.1. ■

4.3 Proof of Theorem 3.3 :

We use the change of probability (28):

$$\mathbb{E}(M_t^2) = \mathbb{E}^{\updownarrow}(M_t)\,, \tag{41}$$

and the spine decomposition (see page 5 for a discussion of this method):

$$M_t = c_t + d_t,$$

where

$$c_t := \frac{e^{\rho t}}{h(0)} h\left(vt + \log(|\Pi_1^\dagger(t)|)\right)\ |\Pi_1^\dagger(t)| \tag{42}$$

and

$$d_t := \frac{e^{\rho t}}{h(0)} \sum_{i=2}^{\infty} h\left(vt + \log\left(|\Pi_i^\dagger(t)|\right)\right)\ |\Pi_i^\dagger(t)|\,. \tag{43}$$

The asymptotic behaviors of c_t and d_t are governed by the two following lemmas.

Lemma 4.1. *Suppose Assumption (A) holds. Under $\mathbb{P}^{\updownarrow}$, $\mathrm{e}^{-(\rho-v)t}c_t$ converges in distribution as $t\to\infty$, to a random variable η with no mass at 0. Moreover there exists $K > 0$ such that*

$$\lim_{t\to\infty} \mathbb{E}^{\updownarrow}(c_t)\mathrm{e}^{-(\rho-v)t} = K\,. \tag{44}$$

Lemma 4.2. *Suppose Assumption (A) holds. If $\rho \neq v$, there exists $L > 0$ such that*

$$\mathbb{E}^{\updownarrow} d_t \leq L \max\{\mathrm{e}^{(\rho-v)t}, 1\}\,. \tag{45}$$

4.3.1 Proof of Theorem 3.3 1)

From (41), it is enough to prove that $\lim_{t\to\infty} \mathbb{E}^{\updownarrow}(M_t) < \infty$. Now, by (44), we have $\lim_{t\to\infty} \mathbb{E}^{\updownarrow}(c_t) = 0$ and by (45), we have $\sup_t \mathbb{E}^{\updownarrow}(d_t) < \infty$. ■

4.3.2 Proof of Theorem 3.3 2) a)

The method is now classic, (see for instance [22]) and uses a decomposition which may be found e.g. in Durrett [16] p. 241. It will be used also in the proof of Theorem 3.4 below. We only need to prove that $\mathbb{P}^{\updownarrow}(\limsup M_t = \infty) = 1$.

We have the obvious lower bound

$$M_t \geq c_t$$

For $v < \rho$, Lemma (4.1) yields $\lim c_t = \infty$ in $\mathbb{P}^{\updownarrow}$ probability, or in other words $\mathbb{P}^{\updownarrow}(\limsup_t c_t = \infty) = 1$ which implies $\mathbb{P}^{\updownarrow}(\limsup_t M_t = \infty) = 1$, hence $\mathbb{P}(\lim_t M_t = 0) = 1$. ■

4.3.3 Proof of Theorem 3.3 2 b)

It is a straightforward consequence of (41) and Lemma 4.1 and 4.2. ■

4.3.4 Proof of Lemma 4.1 :

From the definition (42) of c_t, we see that the distribution under $\mathbb{P}^{\updownarrow}$ of the process $\left(\mathrm{e}^{-(\rho-v)t}c_t, t \geq 0\right)$ is the same as the distribution under $\mathbf{P}^{\updownarrow}_{\log(1/a)}$ of $\left(h(0)^{-1}W^{(-\rho)}(Y_t)\mathrm{e}^{Y_t}\,\mathbf{1}_{\{t<T\}}, t \geq 0\right)$. Under the latter probability, the stopping time T (defined in (24)) is a.s. infinite and from Theorem 2.3, Y is positive-recurrent, it converges in distribution and the limit has no mass in 0. Since the function $y \mapsto W^{(-\rho)}(y)\mathrm{e}^y$ is continuous, it is bounded on the compact support of the distribution of Y_t, and $y_t = \mathbb{E}^{\updownarrow}\left(c_t \mathrm{e}^{-(\rho-v)t}\right)$ has a positive limit. ■

4.3.5 Proof of Lemma 4.2 :

We start from the definition of d_t decomposing the time interval $[0, t]$ into pieces $[k-1, k[$ and splitting the sum (43) according to the time where the fragment separates from 1:

$$h(0)\mathrm{e}^{-\rho t} d_t = \sum_k S_k \tag{46}$$

with

$$S_k(t) = \sum_{i \in \mathcal{I}_k} h(vt + \log|\Pi_i^{\dagger}(t)|)|\Pi_i^{\dagger}(t)|$$

where $\mathcal{I}_k$ is the set of $i \geq 2$ such that the block $\Pi_i(t)$ separates at some instant $r \in \mathcal{D}_1 \cap [k-1, k[$. The block after the split which contains 1 is $\Pi_1(r)$. Thus, there is some index $\ell \geq 2$ such that $\Pi_i(t) \subseteq \pi_\ell(r) \cap \Pi_1(r-)$. Then, at time k, $\pi_\ell(r) \cap \Pi_1(r-)$ is partitioned into $\Pi_j(k), j \in \mathcal{J}_{\ell,r}$ where $\mathcal{J}_{\ell,r}$ is some set of indices measurable with respect to $\mathcal{G}(k)$. Conditionally upon $\mathcal{G}(k)$, the partition $(\Pi_i(t), i \in \mathcal{I}_k)$ can be written in the form $\tilde{\Pi}^{(j)}(t-k)_{|\Pi_j(k)}, j \in \mathcal{J}_k$, where

- $\mathcal{J}_k$ is some set of indices $\mathcal{G}(k)$-measurable

• $(\tilde{\Pi}^{(j)})_{j\in\mathbb{N}}$ is a family of i.i.d. homogeneous fragmentations distributed as Π under $\mathbb{P}$ (see Lemma 3.2).

As a consequence:

$$\underset{i\in\mathcal{I}_k}{\cup}\Pi_i(t) = \cup_{j\in\mathcal{J}_k}\tilde{\Pi}^{(j)}(t-k)_{|\Pi_j(k)}\,, \tag{47}$$

with the slight abuse of notation by which we write a union of partitions instead of the union of the blocks of these partitions, and for all $m\in\mathbb{N}$

$$|\tilde{\Pi}^{(j)}_m(t-k)_{|\Pi_j(k)}| = |\tilde{\Pi}^{(j)}_m(t-k)||\Pi_j(k)|. \tag{48}$$

Now, we have to take into account the killings.

Let us call "good fragment"a fragment which satisfies the constraint of non-killing all along its history up to time t. We can decompose

$$S_k(t) = \sum_{j\in\mathcal{J}_k}|\Pi_j(k)|\left(\sum_m h\left(vt+\log(|\tilde{\Pi}^{(j)}_m(t-k)||\Pi_j(k)|)\right)|\tilde{\Pi}^{(j)}_m(t-k)|\mathbb{1}_{j,m,k}\right)$$

where $\mathbb{1}_{j,m,k}=1$ if and only if $\tilde{\Pi}^{(j)}_m(t-k)_{|\Pi_j(k)}$ is a good fragment. If $\Pi_j(k)$ is a good fragment, we define

$$\widetilde{M}^j_{t-k} := \mathrm{e}^{-\rho(k-t)}\sum_m\frac{h(vt+\log(|\tilde{\Pi}^{(j)}_m(t-k)||\Pi_j(k)|))}{h(vk+\log(|\Pi_j(k)|)}|\tilde{\Pi}^{(j)}_m(t-k)|\mathbb{1}_{j,m,k}\,.$$

From the definition of h in (25), the process $\left(\widetilde{M}^j_{t-k}\right)_{t\geq k}$ is a $(\mathcal{G}(s))_{s\geq k}$ martingale with respect to $\mathbb{P}$, distributed as $(M_t)_{t\geq 0}$.

Denoting $\mathbb{1}_{j,k}=1$ if and only if $\Pi_j(k)$ is a good fragment, from Lemma 3.2 we have

$$\mathbb{E}^{\updownarrow}\left(S_k(t)|\mathcal{G}(k)\right) = \mathrm{e}^{\rho(k-t)}\sum_{j\in\mathcal{J}_k}|\Pi_j(k)|h(vk+\log(|\Pi_j(k)|)\mathbb{1}_{j,k}\,.$$

Now again by the definition of h and its continuity, there exists $C_3>0$ such that

$$h(vk+\log(|\Pi_j(k)|)\leq C_3\mathbb{1}_{vk+\log(|\Pi_j(k)|\in(\log a,\log b)}$$

and

$$\mathbb{E}^{\updownarrow}\left(S_k(t)|\mathcal{G}(k)\right)\leq C_3\mathrm{e}^{(\rho-v)k}\mathrm{e}^{-\rho t}\sum_{j\in\mathcal{J}_k}\mathbb{1}_{j,k}\,.$$

It is clear that the only terms that contribute to the above sum correspond to good fragments at time k which were dislocated from good $\Pi_1(k-1)$ during $[k-1,k[$. Since the fragmentation is conservative, there were at most be^v/a such dislocations during that time, which yields:

$$\mathbb{E}^{\updownarrow}\left(S_k(t)|\mathcal{G}(k)\right)\leq C_3be^va^{-1}\mathrm{e}^{(\rho-v)k}\mathrm{e}^{-\rho t}\,.$$

Coming back to (46) we get

$$\mathbb{E}^{\updownarrow}(d_t)\leq C_4\sum_{k=1}^{\lfloor t\rfloor}\mathrm{e}^{(\rho-v)k}\,,$$

for some constant $C_4>0$. In other words, for all $v\neq\rho$ there exists $L>0$ such that

$$\mathbb{E}^{\updownarrow}(d_t)\leq L\max(\mathrm{e}^{(\rho-v)t},1)\,,$$

which proves (45), hence Lemma 4.2. ■

4.4 Proof of Theorem 3.4

Let us recall the definition of the function

$$d(p,\alpha) = (1+\alpha)\kappa(p) - \kappa\big((1+\alpha)(p+1)-1\big)$$

and let us look at its sign. We have $d(p,0)=0$ and the derivative of $d(p,\alpha)$ in $\alpha=0$ is

$$\kappa(p) - (p+1)\kappa'(p) \begin{cases} <0 & \text{if } p<\bar{p}, \\ =0 & \text{if } p=\bar{p}, \\ >0 & \text{if } p>\bar{p}\,. \end{cases} \tag{49}$$

If $p=\bar{p}$, the second derivative in $\alpha=0$ is $-(\bar{p}+1)\kappa''(\bar{p})>0$ since κ is concave. This ensures that, in any case, there exists $\alpha_0(p)>0$ such that, for every $\alpha\in(0,\alpha_0(p))$

$$d(p,\alpha) \begin{cases} <0 & \text{if } p<\bar{p}, \\ >0 & \text{if } p\geq\bar{p}\,. \end{cases} \tag{50}$$

1) In the proof of Theorem 2 of [7] p.406-407, Bertoin gave the estimate:

$$\mathbb{E}\sup_{0<s\leq t} |\,M_s^{(p)}\,|^{1+\alpha} \leq K_\alpha\, c(p,\alpha)\int_0^t \exp\left(d(p,\alpha)s\right)\,\mathrm{d}s \tag{51}$$

where K_α is a universal constant depending only on α, and

$$c(p,\alpha) = \int_{\mathcal{S}^*} |\sum_{i=1}^{\infty}(x_i^{p+1}-x_i)\,|^{1+\alpha}\ \nu(\mathrm{d}x) < \infty$$

for every $p>\underline{p}$ and $\alpha\in[0,\alpha_1(p)]$ for some $\alpha_1(p)>0$. From (50) above, the integral on the RHS of (51) is then uniformly bounded in t.

2) a) The martingale is bounded by below by the contribution of the spine:

$$M_t^{(p)} \geq \mathrm{e}^{t\kappa(p)}|\Pi_1(t)|^{p+1} = \exp\{t\kappa(p)-(p+1)\xi_t\}\,.$$

As an easy consequence of Lemma 3.1 , we see that under $\mathbb{P}^{(p)}$, the Lévy process $\big(\kappa(p)t-(p+1)\xi_t\big)_{t\geq0}$ has mean $\kappa(p)-(p+1)\kappa'(p)$ which is nonegative from (49) since $p\geq\bar{p}$. We get successively $\mathbb{P}^{(p)}(\limsup_{t\to\infty}(\kappa(p)t-(p+1)\xi_t)=\infty)=1$, hence $\mathbb{P}^{(p)}(\limsup M_t^{(p)}=\infty)=1$, and $\mathbb{P}(\lim M_t^{(p)}=0)$ (see section 4.3.2).

2 b) The only point which remains to prove is (30), and it is a consequence of (51) and (50).

5 Comparison of limits.

Proof of Proposition 1.3: Let us give a direct proof of the inequality

$$v-\rho(v;a,b)\leq C(v)\,. \tag{52}$$

Fix v, a, b and let $\rho = \rho(v; a, b)$ and $\beta = \log b/a$. By the definition (4) of C, (52) is equivalent to

$$pv - \kappa(p) \geq -\rho \tag{53}$$

for every $p > \underline{p}$. Referring to (23) we recall that $pv - \kappa(p) = \psi(p)$, (the Laplace exponent of the process Y), so that (53) is equivalent to

$$\rho + \psi(p) \geq 0\,. \tag{54}$$

If $\psi(p) \geq 0$, there is nothing to prove since ρ is nonegative by definition. Let us assume $\psi(p) < 0$. If W is the scale function of Y, we have

$$\rho = \inf\{q \geq 0 : W^{(-q)}(\beta) = 0\} = \inf\{q' \geq \psi(p) : W^{(\psi(p)-q')}(\beta) = 0\} - \psi(p)\,.$$

Moreover, if W_p is the scale function of the tilted process of Laplace exponent $\lambda \mapsto \psi(\lambda+p) - \psi(p)$, we have

$$W^{(\psi(p)-q')}(\beta) = e^{px} W_p^{(-q')}(x)$$

([23] p.213 and Lemma 8.4 p.222). This yields

$$\rho + \psi(p) = \inf\{q' \geq \psi(p) : W_p^{(-q')}(\beta) = 0\}\,.$$

Since $W_p^{(-q')}(\beta) > 0$ for $q' \leq 0$, the latter infimum is nonegative, so that (54) holds true, which ends the proof. ■

Remark 5.1. *A consequence of this proposition is that when $v < v_{min}$, we have $\rho(v; a, b) > v$.*

Acknowledgements. We would like to thank an Associate Editor and two anonymous referees for their pertinent comments and suggestions on the first draft.

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